Differential Equations 9/7/2020

HW #3 due Friday @ Midnight EST

Office hours Tue 1:30-2:30pm

Please do poll

linear equation

$$\frac{dy}{dt} = a(t)\,y + f(t)$$

$y' = \underbrace{y \sin(x)}_{a(x)}$

$y' = \underbrace{5x}_{f(x)} + \underbrace{3x}_{a(x)}y$

} correct

$y' = 1 - y^2$

$y' = x\underbrace{e^y}$

} nonlinear

which are in homogeneous?

$$y' = \sin(t)\, y + \cos(t)$$

$$a(t) \cdot y + f(t)$$

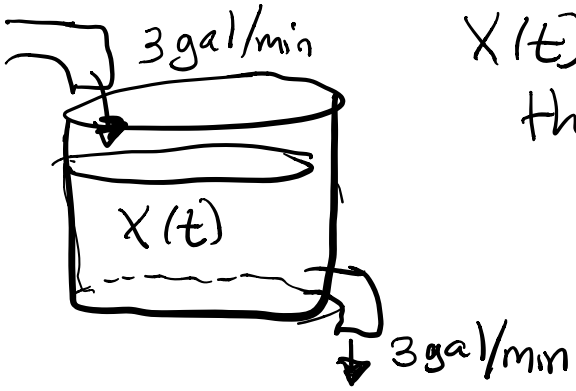
$f(t) = 0$ homogeneous

$$y' = (5t+1)\, y - e^{-t}$$

$y' = a(t) \cdot y$ homogeneous

2.5 Mixing Problems

tank hold 100 gal pure water
at $t=0$ min, solution containing 2 lbs salt per gal at rate 3 gal/min into tank. $t=0$ min a drain is open at bottom → volume remains same (constant)
How much salt is in the tank after 60 min?



$X(t) \equiv$ # lbs of salt in the tank at t

$\frac{dx}{dt}$ = rate at which amount of salt is changing w/ time

Balance Law

$$\frac{dx}{dt} = \text{rate}^{\text{salt}}_{\text{in}} - \text{rate}^{\text{salt}}_{\text{out}}$$

rate in: $\left(\begin{matrix}\text{total}\\ \text{volume in}\end{matrix}\right)\left(\text{[illegible]}/\begin{matrix}\text{salt}\\ \text{concentration}\end{matrix}\right)$

$= (3\text{ gal/min})(2\text{ lb/gal})$

$= 6 \text{ lb/min}$

rate out: (Volume rate)(concentration)

concentration at t: $\frac{X(t)}{100}$ lb/gal

$: \left(3 \text{ gal/min}\right)\left(\frac{X(t)}{100}\right) \text{lb/gal}$

$\frac{3X(t)}{100}$ lb/min

$\frac{dx}{dt} = 6 - \frac{3x}{100}$ ← linear inhomogeneous equation

$\frac{dx}{dt} = f(t) + a(t)x$

$f(t) = 6 \qquad a(t) = -\frac{3}{100}$

Solve for homogeneous eq.

$\frac{dX_h}{dt} = \frac{-3}{100} X_h$

$\int \frac{dX_h}{X_h} = -\int \frac{3}{100} dt$

$$\ln|X_n| = -\frac{3}{100}t$$

$$X_n = e^{-3/100t}$$

$$X_n' = -\frac{3}{100}e^{-3/100t}$$

Sub:

$$X(t) = V(t)X_n(t)$$

solve

$$X_n' = -\frac{3}{100}X_n$$

in $\frac{dx}{dt}$

$$\frac{d}{dt}(V \cdot X_n) = 6 - \frac{3}{100}(VX_n)$$

$$VX_n' + V'X_n = 6 - \frac{3}{100}VX_n$$

$$-\frac{3}{100}X_nV + V'X_n = 6 - \frac{3}{100}VX_n$$

$$\frac{dV}{dt} = \frac{6}{X_n}$$

$$X_n = e^{-\frac{3}{100}t}$$

$$\frac{1}{X_n} = e^{\frac{3}{100}t}$$

$$\frac{dV}{dt} = 6e^{3/100t}$$

$$\int dV = 6\int e^{3/100t}\,dt$$

$$V(t) = 6\left(\frac{100}{3}\right)e^{\frac{-3}{100}t} + C$$

$$V(t) = 200e^{\frac{-3}{100}t} + C$$

$$X(t) = V(t) X_n$$

$$= \left(200e^{\frac{-3}{100}t} + C\right)e^{\frac{-3}{100}t}$$

$$\boxed{X(t) = 200 + Ce^{\frac{-3}{100}t}}$$

no salt was in the tank at $t=0$

$X=0$

$$0 = 200 + Ce^{0}$$

$$C = -200$$

$$X(t) = 200 - 200e^{\frac{-3}{100}t}$$

how much salt at $t = 60$ min?

$$X(t=60) = 200 - 200e^{\frac{-3}{100}60} = 167 \text{ lbs salt}$$

Ex 2:

600 gal tank w/ 300 gal pure water

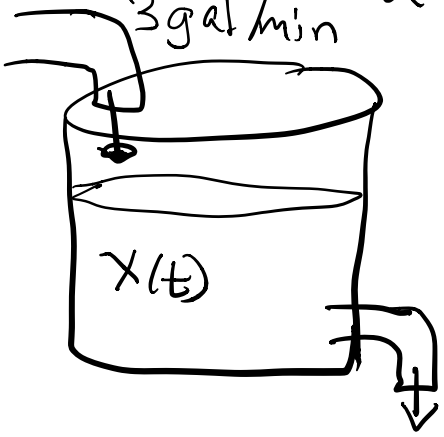
flow in: salt solution 1.5 lb/gal

at rate of 3 gal/min

volume increase at a rate

2 gal/min, leaving at

a rate of 1 gal/min



volume at t

$V(t) = 300 + 2\,^{gal}/_{min} \cdot t$

$= 300 + 2t$ gals

rate in = (volume rate in)(concentration)

$= (3 \text{ gal/min})(1.5 \text{ lb/gal})$

$= 4.5$ lb/min

rate out $= (1 \text{ gal/min}) \frac{X(t)}{300+2t}$ lb/min

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}$$

$$= 4.5 - \frac{x}{300+2t} \text{ lb/min}$$

inhomogeneous linear equation

(1) Find solution to the homogeneous equation.

$$\frac{dx_h}{dt} = \frac{-x_h}{300+2t}$$

$$\int \frac{dx_h}{x_h} = -\int \frac{dt}{300+2t}$$

$$\ln|x_h| = -\frac{1}{2}\ln|300+2t|$$

$$\ln|x_h| = \ln|(300+2t)^{-1/2}|$$

$$x_h = \frac{1}{\sqrt{300+2t}}$$

② use $x = v \cdot x_h$ find v

plug into $\frac{dx}{dt}$

$$\frac{dx}{dt} = 4.5 - \frac{x}{300+2t}$$

$$v'x_h + vx_h' = 4.5 - \left(\frac{1}{300+2t}\right) vx_h$$

aside

$$\begin{cases} x_h = \frac{1}{\sqrt{300+2t}} = (300+2t)^{-1/2} \\ x_h' = -\frac{1}{2}(300+2t)^{-3/2}(2) \end{cases}$$

$$v'x_h + vx_h' = 4.5 - \frac{1}{300+2t} vx_h$$

$$v'\frac{1}{\sqrt{300+2t}} + v\left(-\frac{1}{(300+2t)^{3/2}}\right)$$

$$= 4.5 = \left(\frac{1}{300+2t}\right)V\left(\frac{1}{\sqrt{300+2t}}\right)$$

$$V' \frac{1}{\sqrt{300+2t}} = 4.5$$