Differential Equations

9/16/2020

HW # 4 due Friday @ Midnight EST

HW # 2 solutions up on Canvas

Office Hours tomorrow 4-5pm

Midterm 1 Sept 29 - Oct 1
noon - noon

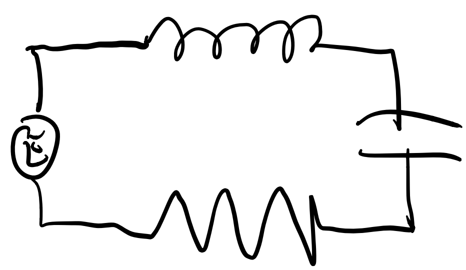
Practice Midterm exams on canvas
closed notes

linear homogeneous Equations with constant coefficients

$$y'' + p(t)y' + q(t)y = 0$$

$$y'' + py' + qy = 0 \qquad \begin{matrix} p \geq 0 \\ q > 0 \end{matrix} \quad \text{constants}$$

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = 0 \qquad \text{RLC equation}$$



$$y'' + \mu y' + \frac{k}{m}y = 0 \qquad \text{unforced vibrating spring}$$

two linearly independent solutions

not constant multiples of each other

fundamental set of solutions

recall for first order
w/ constant coefficients:

$$y' + py = 0$$

- homogeneous
- seperable
- exponential equation

general solution $y(t) = Ce^{-pt}$

Find exp. to 2nd order of the form $y(t) = e^{\lambda t}$

$$y'' + py' + qy = 0$$

Check if solution

$$\frac{d^2}{dt^2} e^{\lambda t} + p\frac{d}{dt} e^{\lambda t} + q e^{\lambda t} = 0$$

$$\lambda^2 e^{\lambda t} + \lambda p e^{\lambda t} + q e^{\lambda t} = 0$$

$$(\lambda^2 + \lambda p + q) e^{\lambda t} = 0$$

Characteristic equation

Characteristic polynominal:

$$\lambda^2 + p\lambda + q = 0$$

Characteristic root λ

then

$y(t) = e^{\lambda t}$ is a solution

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

look at the discriminant:

$p^2 - 4q$:

1. two distinct real roots $p^2 - 4q > 0$
2. two complex roots $p^2 - 4q < 0$
3. one repeated root $p^2 - 4q = 0$

Case 1 Distinct real roots

λ_1, λ_2 distinct real roots,

$y_1 = e^{\lambda_1 t} \quad y_2 = e^{\lambda_2 t}$

are both solutions

since $\lambda_1 \neq \lambda_2$

then $\dfrac{y_1}{y_2} = \dfrac{e^{\lambda_1 t}}{e^{\lambda_2 t}} = \underline{e^{(\lambda_1 - \lambda_2)t}}$

not a constant

linearly independent
if not constant multiples of each other

$$\begin{cases} y_1 = a e^{\lambda_1 t} \\ \underline{y_2} = b e^{\lambda_1 t} = \dfrac{b}{a} y_1 \end{cases}$$

$\dfrac{y_1}{\quad} = \dfrac{a \cancel{e^{\lambda_1 t}}}{\cancel{\lambda t}} = \dfrac{a}{\quad}$

y_2 De D

general solution:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Ex. $y'' - 3y' + 2y = 0$

initial conditions $y(0) = 2$

$y'(0) = 1$

insert $y = e^{\lambda t}$

$$\frac{d^2}{dt} e^{\lambda t} - 3\frac{d}{dt} e^{\lambda t} + 2e^{\lambda t} = 0$$

$$(\lambda^2 - 3\lambda + 2)e^{\lambda t} = 0$$

$$\lambda^2 - 3\lambda + 2 = 0$$

$$(\lambda - 1)(\lambda - 2) = 0$$

$\lambda_1 = 1 \quad \lambda_2 = 2$ characteristic roots

$$y(t) = c_1 e^t + c_2 e^{2t}$$

- $y(0) = 2$:

$$2 = c_1 \cancel{e^0} + c_2 \cancel{e^0}$$

$$\boxed{2 = c_1 + c_2}$$

- $y'(0) = 1$

$$y'(t) = c_1 e^t + 2c_2 e^{2t}$$

$$1 = c_1 \cancel{e^0} + 2c_2 \cancel{e^0}$$

$$\boxed{1 = c_1 + 2c_2}$$

$$
\begin{array}{rl}
 & 2 = c_1 + c_2 \\
- & 1 = c_1 + 2c_2 \\
\hline
 & 1 = 0 - c_2
\end{array}
$$

$\boxed{-1 = c_2}$

$\boxed{c_1 = 3}$

$$\boxed{y(t) = 3e^{t} - e^{2t}}$$

particular solution

Case 2: complex roots

$$y'' + py' + qy = 0$$

$$p^2 - 4q < 0$$

roots λ_1 and λ_2

complex conjugates of eache other

$\lambda = \lambda_1 = a + ib$ — a & b are real constants.

$\bar{\lambda} = \lambda_2 = a - ib$

complex conjugate

$i \to -i$

$$\begin{cases} z(t) = e^{(a+ib)t} = e^{at}e^{ibt} \\ \bar{z}(t) = e^{(a-ib)t} = e^{at}e^{-ibt} \end{cases}$$

recall Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z(t) = e^{at}\underline{e^{ibt}} = e^{at}\left[\cos(bt) + i\sin(bt)\right]$$

$$\bar{z}(t) = e^{at}e^{-ibt} = e^{at}\left[\cos(bt) - i\sin(bt)\right]$$

are these linearly independent?

$$\frac{z(t)}{\bar{z}(t)} = \frac{\cancel{e^{at}}e^{ibt}}{\cancel{e^{at}}e^{-ibt}} = e^{2ibt}$$

not constant multiples

So yes they are linearly independent

$$y(t) = c_1 \underline{z(t)} + c_2 \underline{\bar{z}(t)}$$

rewrite:

$$\left.\begin{aligned} z(t) &= y_1(t) + i y_2(t) \\ \bar{z}(t) &= y_1(t) - i y_2(t) \end{aligned}\right\}$$

we want real valued solutions.

$$y_1 = Re(z(t)) = Re\left[e^{at}(\cos bt + \cancel{i \sin bt})\right]$$

$$= \underline{e^{at}\cos bt}$$

$$y_2 = Im(z(t)) = \underline{Im}\left[e^{at}(\cancel{\cos bt} + i\sin bt)\right]$$

$$= e^{at}\sin(bt)$$

$$y_1 = \frac{1}{2}\left[z(t) + \bar{z}(t)\right]$$

$$= \frac{1}{2}\left[e^{at}(\cos bt + i\sin bt) + e^{at}(\cos bt - i\sin bt)\right]$$

$$y_2 = \frac{1}{2i}\left[z(t) - \bar{z}(t)\right]$$

$$= \frac{1}{2i}\left[e^{at}(\cancel{\cos bt} + i\sin bt) - e^{at}(\cancel{\cos bt} - i\sin bt)\right]$$

$$y_1(t) = \frac{1}{2}\left[z(t) + \bar{z}(t)\right]$$

$$y_2(t) = \frac{1}{2i}\left[z(t) - \bar{z}(t)\right]$$

general solution:

$y(t) = A_1 y_1 + A_2 y_2$

$$\boxed{y(t) = A_1 e^{at}\cos bt + A_2 e^{at}\sin bt}$$

A_1, A_2 real constants

$\lambda \quad \bar{\lambda}$

$$\boxed{\begin{array}{l} \text{Ex. } y'' + 2y' + 2y = 0 \\ \quad y(0) = 2, \; y'(0) = 3 \end{array}}$$

Characteristic equation

plug in $y = e^{\lambda t}$

$$\frac{d^2}{dt^2} e^{\lambda t} + 2\frac{d}{dt} e^{\lambda t} + 2e^{\lambda t} = 0$$

$$(\lambda^2 + 2\lambda + 2)e^{\lambda t} = 0$$

Characteristic polynomial:

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(1)(2)}}{2}$$

$$\lambda = \frac{-2 \pm \sqrt{-4}}{2}$$

$$\lambda = \frac{-2 \pm 2i}{2}$$

$$\lambda = -1 \pm i$$

$$\lambda = a + ib \qquad \left.\begin{aligned} a &= -1 \\ b &= 1 \end{aligned}\right\}$$

$$\bar{\lambda} = a - ib$$

$$y_1(t) = e^{at} \cos bt$$

$$\boxed{y_1(t) = e^{-t}\cos t}$$

$$y_2(t) = e^{at}\sin bt$$

$$\boxed{y_2(t) = e^{-t}\sin t}$$

general solution:

$$y(t) = A_1 y_1 + A_2 y_2$$

$$\boxed{y(t) = A_1 e^{-t}\cos t + A_2 e^{-t}\sin t}$$