Differential Equations

9/18/2020

HW # 4 due tonight @ midnight

HW # 5 is up

Midterm Sept 29 noon - Oct 1 noon EST

practice exams on Canvas

Please do poll

Linear homogeneous Equations w/ constant coefficients

$$y'' + py' + qy = 0$$

$$\lambda^2 + p\lambda + q = 0$$

$$\boxed{\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}}$$

$$\sqrt{p^2 - 4q}$$

1. two real distinct roots
   $p^2 - 4q > 0$
2. two distinct complex root (complex conjugates)
   $p^2 - 4q < 0$
3. One repeated root $p^2 - 4q = 0$

Case 2 Continued Complex Roots

$$y(t) = A_1 e^{at}\cos bt + A_2 e^{at}\sin bt$$

A_1, A_2 real constants

$$\lambda = \underline{a} \pm i\,\underline{b}$$

ex. $y'' + 2y' + 2y = 0$

Charastistic equation

$y = e^{\lambda t}$

$$\frac{d^2}{dt^2} e^{\lambda t} + 2\frac{d}{dt} e^{\lambda t} + 2e^{\lambda t} = 0$$

$$\underbrace{(\lambda^2 + 2\lambda + 2)}\,e^{\lambda t} = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{4-8}}{2}$$

$$= \frac{-2 \pm \sqrt{-4}}{2}$$

$$\lambda = -1 \pm i$$

$$\lambda = a \pm ib \qquad a = -1$$
$$b = 1$$

$$y(t) = A_1 e^{at} \cos bt + A_2 e^{at} \sin bt$$

$$y(t) = A_1 e^{-t} \cos t + A_2 e^{-t} \sin t$$

general solution

$$y(0) = 2 \qquad y'(0) = 3$$

$$2 = A_1 \overset{1}{\cancel{e^0}} \overset{1}{\cancel{\cos(0)}} + \cancel{A_2 e^0 \sin(0)}$$

$$2 = A_1$$

$$y'(t) = \frac{d}{dt}\left(A_1 e^{-t} \cos t\right) + \underline{\frac{d}{dt}\left(A_2 e^{-t} \sin t\right)}$$

$$= A_1 e^{-t}(-\sin t) + A_1(-1)e^{-t}\cos t$$

$$+ A_2 e^{-t}\cos t + A_2(-1)e^{-t}\sin t$$

$$= e^{-t}\left(-A_1 \sin t - A_1 \cos t + A_2 \cos t - A_2 \sin t\right)$$

$$y'(0) = 3$$

$$3 = \overset{1}{e^{0}}\left(-\overset{0}{\cancel{A\sin(0)}} - \underline{A_1}\,\overset{1}{\cos(0)} + \underline{A_2}\,\overset{1}{\cos(0)} - A_2\,\overset{0}{\cancel{\sin(0)}}\right)$$

$$3 = -A_1 + A_2$$

$$3 = -2 + A_2$$

$$5 = A_2$$

$$\boxed{y(t) = 2e^{-t}\cos t + 5e^{-t}\sin t}$$

<u>Case 3</u> Repeated Roots

Characteristic equation:

$$\lambda^2 + p\lambda + q = 0$$

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

$$p^2 - 4q = 0$$

$$\lambda = -\frac{p}{2}$$

$p = 4q$

$$\boxed{\frac{p^2}{4} = q}$$

another solution to add to this:

$$y_1(t) = e^{\lambda t} = e^{-p/2\, t}$$

want a solution that's <u>not</u> a constant multiple

Search for a second solution.

$$y_2(t) = v(t)\, y_1(t) = v(t)\, e^{-pt/2}$$

$v(t)$ is some non constant function of t

$$y'' + py' + qy = 0$$

$$y'' + py' + \frac{p^2}{4} y = 0$$

$$y_2''(t) = \frac{d}{dt}\left(y_2'(t)\right)$$

$$= \frac{d}{dt}\left[e^{-pt/2} v' - \frac{p}{2} v e^{-pt/2}\right]$$

$$= -\frac{p}{2} e^{-pt/2}\left[v' - \frac{p}{2} v\right]$$

$$+ e^{-pt/2}\left[v'' - \frac{p}{2} v'\right]$$

$$= e^{-pt/2}\left[-\frac{p}{2}v' + \frac{p^2}{4}v + v'' - \frac{p}{2}v'\right]$$

$$= e^{-pt/2}\left[v'' - pv' + \frac{p^2}{4}v\right]$$

plug into differential equation

$$y'' + py' + \frac{p^2}{4}y = 0$$

$$e^{-pt/2}\left[v'' - pv' + \frac{p^2}{4}v\right]$$

$$+ p\left[e^{-pt/2}\left(v' - \frac{p}{2}v\right)\right]$$

$$+\frac{p^2}{4} v e^{-pt/2} = 0 \Big)$$

$$e^{-pt/2} v'' = 0$$

isn't zero

must be zero

what is a function that is zero after 2 derivatives

$$v(t) = t$$

$$v'(t) = 1$$

$$v''(t) = 0$$

$y_2(t) = v(t)e^{\lambda t} = t\,e^{\lambda t}$

general solution is

$y(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}$

$$\boxed{y(t) = (C_1 + C_2 t)\,e^{\lambda t}}$$

Ex. Find the general solution to

$y'' - 2y' + y = 0 \qquad y(0) = 2$

$\qquad\qquad\qquad\qquad y'(0) = -1$

$\lambda^2 - 2\lambda + \lambda = 0$

$(\lambda - 1)(\lambda - 1) = 0$

$\lambda = 1$ this is a double root

double root

So: $y_1(t) = e^{\lambda t} = e^{t}$

$y_2(t) = t \cdot e^{\lambda t} = te^{t}$

are solutions

general solution: plug in $\lambda = 1$

$$y(t) = C_1 e^{t} + C_2 t e^{t}$$

$$2 = C_1 e^{0} + \cancel{C_2(0)e^{0}}$$

$$2 = C_1$$

$$y'(t) = C_1 e^{t} + C_2(te^{t} + e^{t})$$

$$-1 = C_1 e^{0} + C_2(\cancel{0e^{0}} + e^{0})$$

$$-1 = 2 + C_2$$

$$-3 = C_2$$

$$\boxed{y(t) = 2e^t - 3te^t}$$

Summary:

2nd order linear homogeneous equation

$y'' + py' + qy = 0$

constant coefficients p, q

Characteristic equation

$\lambda^2 + p\lambda + q = 0$

3 possible cases for our solutions.

- $p^2 - 4q > 0$, characteristic equation has 2 real roots λ_1, λ_2

the fundamental set of solutions is

$$y_1(t) = e^{\lambda_1 t} \quad y_2(t) = e^{\lambda_2 t}$$

general solution

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

- $p^2 - 4q < 0$, characteristic equation has 2 complex conjugate roots:

  $\lambda = a \pm ib$, fundamental set of solutions:

$$y_1(t) = e^{at} \cos bt$$

$$y_2(t) = e^{at} \sin bt$$

general solution:

$$y(t) = A_1 e^{at} \cos bt + A_2 e^{at} \sin bt$$

- $p^2 - 4q = 0$, characteristic equation has one repeated root λ
fundamental set of solutions:
$y_1(t) = e^{\lambda t}$ $y_2(t) = t e^{\lambda t}$

general:
$$y(t) = (C_1 + C_2 t) e^{\lambda t}$$