Differential Equations

9/21/20

HW #5 due Friday, HW3 solutions up today

Office hours Tues 1:30p - 2:30 pm EST

Thurs 4pm - 5pm EST

extra Mon 12:30 - 2:30 EST

Midterm 1 Starts noon EST September 29

ends noon October 1

48 hours

No notes, no internet,

no working together, no calculator

Likely:

1. 1st order IVP (separation of variables)
2. Mass/Spring, Mixing, Circuit
3. 2nd order
4. Something from this week.

Bonus question

review: interval of existence

linearly independent

Fundamental set of solutions

Harmonic Motion

equation for motion of a vibrating Spring

$$my'' + \mu y' + ky = F(t)$$

$m =$ mass

$\mu =$ damping constant

$k =$ Spring constant

$$LI'' + RI' + \frac{1}{C}I = \frac{dE}{dt}$$

$R \sim$ damping

$\frac{1}{C} \sim$ spring constant

$L \sim$ mass

$\frac{dE}{dt} \sim$ external force

Divide by leading coefficient

$$y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t)$$

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{1}{L} \frac{dE}{dt}$$

$$c = \frac{\mu}{2m} = \frac{R}{2L}$$ $c \geq 0$ damping constant

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{1}{LC}}$$ $\omega_0 > 0$ natural freq.

$$f(t) = \frac{F(t)}{m} = \frac{1}{L} \frac{dE}{dt}$$ forcing term

$$x = y = I$$

$$\boxed{x'' + 2c x' + \omega_0^2 x = f(t)}$$

Harmonic motion

First unforced harmonic oscillator
$f(t) = 0$

$$x'' + 2cx' + \omega_0^2 x = 0$$

$c = 0$ no damping

$$x'' + \omega_0^2 x = 0$$ Simple harmonic motion

SHM

$$\lambda^2 + \omega_0^2 = 0$$

$$\lambda = \pm i\omega_0$$ complex roots

$a = 0$
$b = \omega_0$

recall

$$\lambda = a \pm ib$$

Solution:

$$x(t) = e^{at}\left[A_1 \cos bt + A_2 \sin bt\right]$$

$$\boxed{x(t) = A_1 \cos \omega_0 t + A_2 \sin \omega_0 t}$$

$$T = \frac{2\pi}{\omega_0} \quad \text{period of oscillations}$$

$$x(t+T) = x(t) \quad \text{for all } t$$

Ex. Simple harmonic motion with $\omega_0 = 4$ and initial condition $x(0) = 1$

$$x'(0) = 0$$

$$x(t) = A\cos 4t + B\sin 4t$$

$$1 = x(0) = A\,\overset{1}{\cancel{\cos(0)}} + B\,\overset{0}{\cancel{\sin(0)}}$$

$$1 = A$$

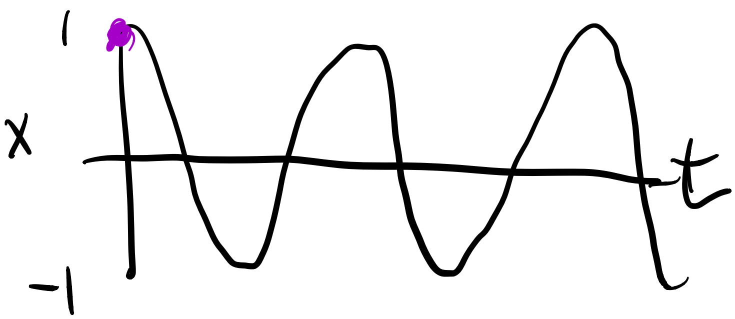
$$x'(t) = -4A\sin 4t + 4B\cos 4t$$

$$0 = -4A\,\overset{0}{\cancel{\sin(0)}} + 4B\cos(0)$$

$0 = B$

$$\boxed{x(t) = \cos 4t}$$

particular solution

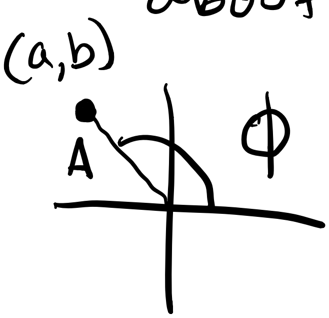


$x(t) = a\cos 4t$

$a = 1$

$b = 0$

Find a new form for general solutions → more insight about meaning of solution



Vector (a,b), length A and angle ϕ

$a = A\cos\phi \qquad b = A\sin\phi$

$$\boxed{x(t) = a\cos\omega t + b\sin\omega t}$$

$x(t) = A\cos\phi\cos\omega t + A\sin\phi\sin\omega t$

using trig property

$$\boxed{\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2}$$

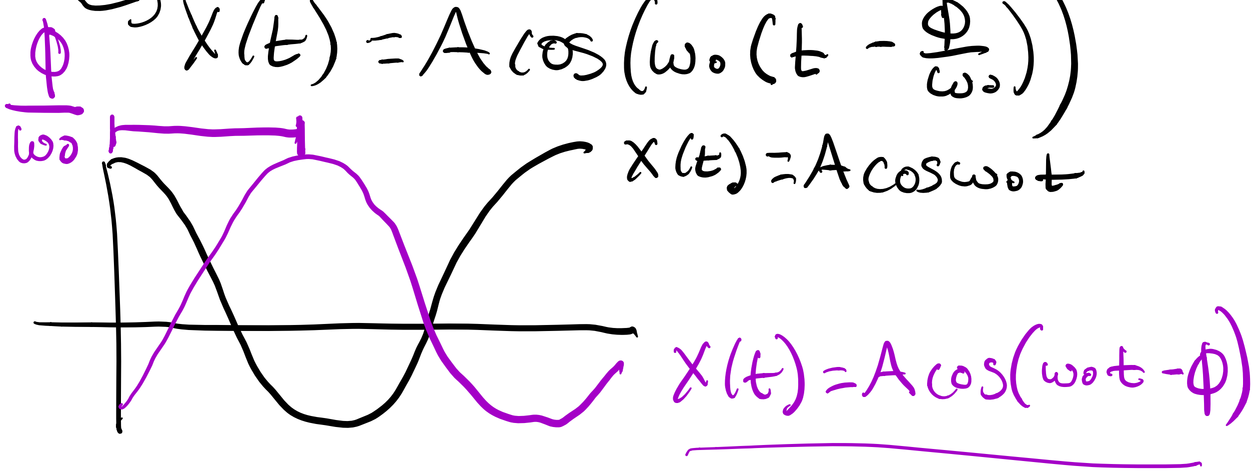
$x_1 = \omega_0 t$

$x_2 = \phi$

$$\boxed{x(t) = A\cos(\omega_0 t - \phi)}$$

ϕ phase shift of our solution

$$x(t) = A\cos\left(\omega_0\left(t - \frac{\phi}{\omega_0}\right)\right)$$



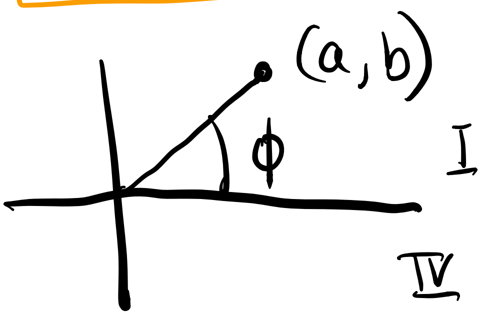
$$x(t) = a\cos\omega_0 t + b\sin\omega_0 t$$

what are A and ϕ

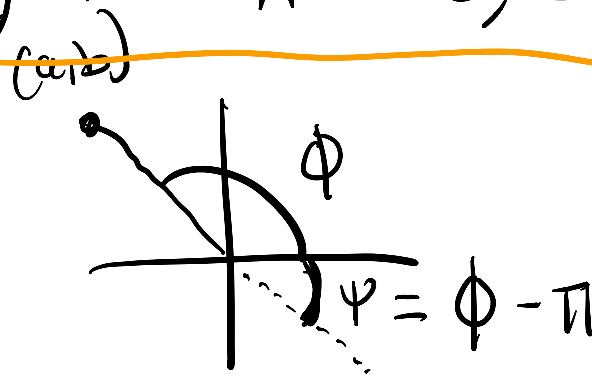
$$A^2 = a^2 + b^2 \quad \text{and} \quad \tan\phi = \frac{b}{a}$$

Solving for ϕ, be careful about quadrants:

$$\boxed{\phi = \begin{cases} \arctan b/a & \text{if } a > 0 \\ \arctan\left(\frac{b}{a}\right) + \pi & \text{if } a < 0,\ b > 0 \\ \arctan\left(\frac{b}{a}\right) - \pi & \text{if } a < 0,\ b < 0 \end{cases}}$$

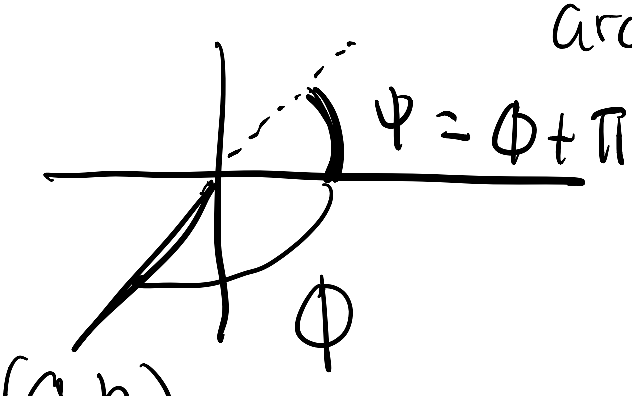


$a > 0$



$a < 0$ &
$b > 0$

$$\arctan\frac{b}{a} = \psi$$



$(\omega, 0)$

Damped harmonic motion

if $c > 0$

$$x'' + 2cx' + \omega_0^2 x = 0$$

$$\lambda^2 + 2c\lambda + \omega_0^2 = 0$$

$$\left.\begin{aligned} \lambda_1 &= -c - \boxed{\sqrt{c^2 - \omega_0^2}} = \\ \lambda_2 &= -c + \sqrt{c^2 - \omega_0^2} = \end{aligned}\right\} -c \pm i\sqrt{}$$

three cases depending on discriminant

1. $c < \omega_0$ underdamped case

   Roots complex conjugates

   $$x(t) = e^{-ct}\left[A_1 \cos \omega t + C_2 \sin \omega t\right]$$

   $$\omega = \sqrt{\omega_0^2 - c^2}$$

2. $c > \omega_0$ overdamped case

2 distinct real roots

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

roots are negative

$$\lambda_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$$

solution → decaying as $t \to \infty$

3. $c = \omega_0$ Critically damped case

roots are real & repeated

$$\lambda = -c \pm \sqrt{c^2 - c^2}$$

$$\lambda = -c$$

$$x(t) = A_1 e^{-ct} + A_2 t e^{-ct}$$

$t \to \infty$

decaying

Review

general harmonic motion:

$$x'' + 2c x' + \omega_0^2 x = 0$$

unforced

when $c = 0$

SHM: no damping

solutions:

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t$$

with damping: 3 cases

1. Underdamped $c < \omega_0$

$$x(t) = e^{-ct}\left[A_1\cos\omega t + A_2\sin\omega t\right]$$

$$\omega = \sqrt{\omega_0^2 - c^2}$$

2. $c > \omega_0$ overdamped

$$x(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}$$

$\lambda_{1,2}$ real, neg

3. $c = \omega_0$ critically damped

$$x(t) = A_1 e^{-ct} + A_2 t e^{-ct}$$