Differential Equations 9/4/2020

HW #2 due tonight @midnight EST

Today last day Add/Drop

Class on Monday (Sorry :( )

Please do poll

If using matlab for HW#2, upload

.m & .mat files

2.4 Linear Equations

First-order linear equation

$$x' = a(t)x + f(t)$$

inhomogeneous

$f(t) = 0 \quad x' = a(t)x$ - can it be solved by separation of variable

homogeneous

linear	nonlinear
$x' = \sin(t)x$	$x' = t\sin(x)$
$y' = e^{2t}y + \cos t$	$y' = y \cdot y'$
$x' = (3t+2)x + t^2 - 1$	$y' = 1 - y^2$

$y = mx + b$

$x' = a(t)x + f(t)$

$$\frac{dx}{dt} = a(t)\,x \quad \text{separable}$$

$$\frac{dx}{x} = a(t)\,dt$$

$$x(t) = A e^{\int a(t)\,dt}$$

Ex $\quad x' = \sin t \cdot x$

$$\int \frac{dx}{x} = \int \sin(t)\,dt$$

$$\ln|x| = -\cos t + C$$

$$|x(t)| = e^{C} e^{-\cos t}$$

$$x(t) = A e^{-\cos t}$$

Inhomogeneous equation

Newton's Law of cooling

$$T' = -k(T - A)$$

$$T' = -kT + kA$$

$$\underline{T' + kT} = kA$$

$$e^{kt}(T' + kT)$$

$$T'e^{kt} + ke^{kt}T$$

product rule

$$[e^{kt}T]'$$

$$[e^{kt}T]' = kAe^{kt}$$

$$\frac{d}{dt}\left(e^{kt}T\right) = kAe^{kt}$$

$$e^{kt}T = \int kAe^{kt}\,dt$$

$$\frac{e^{kt}T}{e^{kt}} = \frac{Ae^{kt}+C}{e^{kt}}$$

$$T(t) = A + C\frac{1}{e^{kt}}$$

$$\boxed{T(t) = A + Ce^{-kt}}$$

$$x' - ax = f$$

~~instead~~ of e^{kt}, use

$u(t)$ integrating factor

$$\underbrace{u(t)x' - au(t)x} = u(t)f(t)$$

$$\left[u(t)x\right]' = u(t)f(t)$$

$$u(t)x(t) = \int u(t)f(t)\,dt$$

$$\boxed{x(t) = \frac{1}{u(t)}\int u(t)f(t)\,dt}$$

Key: find $u(t)$

$$\underbrace{(ux)'}= u(x'-ax)$$

$$\cancel{ux'}+u'x = \cancel{ux'}-aux$$

$$u'\cancel{x} = -au\cancel{x}$$

$$u' = -a\cdot u$$

linear homogeneous equation

Soln:

$$u(t) = e^{-\int a(t)\,dt}$$

no const. A

set $A = 1$

Method #1

for non-homogeous linear equation

$x' = ax + f$ solve by:

1. Rewrite as

$$x' - ax = f$$

2 multiply by integrating factor

$$u(t) = e^{-\int a(t)dt}$$

we get

$$(ux)' = u(x' - ax) = uf$$

3. Integrate to ge

$$u(t)x(t) = \int u(t)f(t)dt + C$$

4. Solve for $x(t)$

Ex $x' = x + e^{-t}$

1. $x' - x = \underline{e^{-t}}$

$x' - a(t)x$

$a(t) = 1$

2. $u(t) = e^{-\int dt} = e^{-t}$

3. $(e^{-t}x)' = e^{-t}e^{-t}$

$(e^{-t}x)' = e^{-2t}$

$e^{-t}x = \underline{\int e^{-2t}dt} + C$

$$\frac{\cancel{e^{-t}}x}{\cancel{e^{-t}}} = \frac{-\frac{1}{2}e^{-2t} + C}{e^{-t}}$$

$$\boxed{x = -\frac{1}{2}e^{-t} + Ce^{t}}$$

Second Method

$y' = -2y + 3$ nonhomogeneous

$y_h' = -2y_h$ homogeneous

$y_h(t) = Ce^{-2t}$

replace C with $v(t)$

$y = \underline{v(t)e^{-2t}}$

sub into $y' = -2y + 3$

$\underline{\left(v(t)e^{-2t}\right)'} = -2\left(v(t)e^{-2t}\right) + 3$

$v'e^{-2t} \underline{\cancel{-2ve^{-2t}}} = \underline{\cancel{-2ve^{-2t}}} + 3$

$v'e^{-2t} = 3$

$$v' e^{-2t} = 3$$
$$v' = 3e^{2t}$$
$$v(t) = \frac{3}{2}e^{2t} + C$$

$$y(t) = v(t)\, y_h$$
$$= \left(\frac{3}{2}e^{2t} + C\right) \cdot e^{-2t}$$

$$\boxed{y(t) = \frac{3}{2} + Ce^{-2t}}$$

Method #2 Steps

1. Find the particular solution of homogeneous equation

$$y_h' = a\, y_h$$

Part. Solution: $y_h = e^{\int a(t)\, dt}$

2. Sub $y = v\,y_h$ into the inhomogeneous DE:

$$y' = ay + f$$

to find $v(t)$

3. Write down general solution:

$$y(t) = v(t)\,y_h(t)$$

Variation of parameters

$$x' = x\tan t + \sin t$$

homogeneous:

$$x_h' = x_h \tan t$$

$$\frac{dx_h}{dt} = x_h \tan t$$

$$\int \frac{dx_n}{x_n} = \int \tan t \, dt$$

$$\ln|x_n| = \int \frac{\sin t}{\cos t} dt$$

$u = \cos t$

$du = -\sin t \, dt$

$$\ln|x_n| = \int \frac{du}{u}$$

$$\ln|x_n| = -\ln|u|$$

$$\ln|x_n| = \ln \frac{1}{u}$$

$$x_n = \frac{1}{u} = \frac{1}{\cos t}$$

homogeneous solution

$$x = v x_n = \frac{v}{\cos t}$$

$$x' = x \tan t + \sin t$$

$$X' = x\tan t + \sin t$$

$$\left[\frac{v}{\cos t}\right]' = \frac{v}{\cos t}\tan t + \sin t$$

$$\frac{\cos t\, v' + v\sin t}{\cos^2 t} = \frac{v}{\cos t}\,\frac{\sin t}{\cos t} + \sin t$$

$$\frac{v'\cancel{\cos t}}{\cos^{\cancel{2}} t} + \cancel{\frac{v\sin t}{\cos^2 t}} = \cancel{\frac{v\sin t}{\cos^2 t}} + \sin t$$

$$\frac{v'}{\cos t} = \sin t$$

$$v' = \sin t\cos t$$

$$v(t) = -\frac{\cos^2 t}{2} + C$$

$$X(t) = v(t)\,X_h$$

$$\boxed{X(t) = \left(-\frac{\cos^2 t}{2} + C\right)\frac{1}{\cos t}}$$