Differential Equations

9/14/2020

HW 4 due Friday 9/18 @ midnight EST

Office Hours tomorrow (Tues) 1:30-2:30pm EST

- Reminder Midterm #1 Tues 9/29 - Thur 10/1
  48 hr. noon EST - noon EST

Practice Midterms on canvas → "File" → "Practice Exams"

Please do Poll

$$\int u\,dv = uv - \int v\,du$$

$$\int t\cos t\,dt$$

$$\boxed{\begin{array}{ll} u = t & v = \sin t \\ du = dt & dv = \cos t\,dt \end{array}}$$

Second Order Differential Equations

$y'' = f(t, y, y')$ normal form

$y' = f(t, y)$ recall

generally:

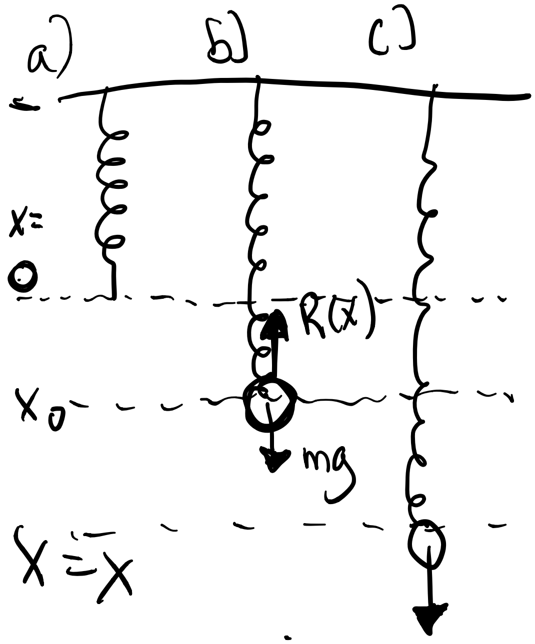
forcing term

$$y'' + p(t)y' + q(t)y = g(t)$$

linear, inhomogeneous

can't separation of variables

if $g(t) = 0$ homogeneous



a) spring w/ no mass
spring-equilibrium → no motion, $x = 0$, $x' = 0$
below x positive

b) attached a weight stretches spring, new equilibrium at
$x = x_0$ $x' = 0$
Spring-mass equilibriums
gravity down, "restorative force/spring force ↑"

$\Sigma F = 0 = R(x_0) + mg = 0$

c) spring is stretched
weight is not in equilibrium
$v = x'$

also damping D (resistance to motion of mass through air/medium)

$D(v)$, if not moving $D=0$

there can also be an external force $F(t)$

ma = total force acting on weight

$$ma = \underbrace{R(x)}_{\text{spring}} + mg + \underbrace{D(v)}_{\text{damping}} + F(t)$$

$a = \ddot{x} \quad v = \dot{x}$

$$\Rightarrow m\ddot{x} = \underline{R(x)} + mg + D(\dot{x}) + F(t)$$

<u>Hooke's Law</u>:

$$R(x) = -Kx \qquad \underset{\text{constant}}{K} > 0$$

acts to decrease displacement of the mass (only valid for small displacements)

$$m\ddot{x} = -Kx + mg + D(\dot{x}) + F(t)$$

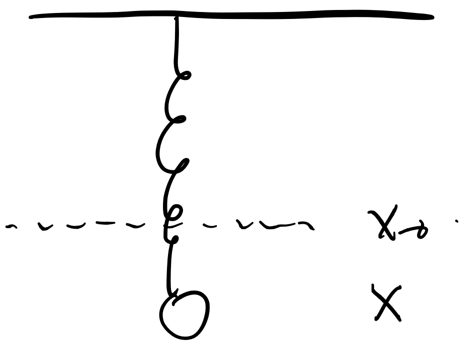
case w/no external Force ($F(t)=0$) and in spring-mass equilibrium

$$x = x_0 \quad , \dot{x} = \ddot{x} = 0$$

$$0 = -kx_0 + mg$$

$$kx_0 = mg \qquad K = \frac{mg}{x_0}$$

$$m\ddot{x} = -K(x - x_0) + D(\dot{x}) + F(t)$$



redefine:

$$y = x - x_0$$

$$\dot{y} = \dot{x}$$

$$\ddot{y} = \ddot{y}$$

$$m\ddot{y} = -Ky + D(\dot{y}) + F(t)$$

$$D(\dot{y}) = -\mu V = -\mu\dot{y}$$

goes against velocity

μ is non negative

often a constant

$$m\ddot{y} + Ky + \mu\dot{y} = F(t)$$

$$m\ddot{y} + Ky = 0$$

$$\boxed{\ddot{y} = -\frac{K}{m}y}$$

$$\sin(At) \xrightarrow{\frac{d}{dt}} A\cos(At) \xrightarrow{\frac{d}{dt}} -A^2\sin(At)$$

$$\cos(At) \longrightarrow -A\sin(At) \longrightarrow -A^2\cos(At$$

general solution

$$y(t) = a\cos\left(\sqrt{\frac{K}{m}}\,t\right) + b\sin\left(\sqrt{\frac{K}{m}}\,t\right)$$

$$\dot{y}$$

$$\ddot{y} = -\frac{K}{m}y$$

$$\sqrt{\frac{K}{m}} = w_0$$ natural frequency

period $\boxed{T = \frac{2\pi}{w_0}}$

$$\nu = f = \text{numerical frequency} = \frac{\text{cycles}}{\text{second}} = \frac{1}{T} = \frac{w_0}{2\pi}$$

General Solution to 2nd Order Diff Eq

$$y'' + p(t)y' + q(t)y = 0$$

→ y_1, y_2 are both solutions to homogeneous equation

also a solution → $\boxed{y = C_1 y_1 + C_2 y_2}$

a linear combination of two solutions with C_1, C_2 constants

* **Any linear combination of two solutions is also a solution**

Ex $y'' - y' - 2y = 0$

if $y_1(t) = e^{-t}$ $y_2(t) = e^{2t}$ are solutions

then $y(t) = C_1 e^{-t} + C_2 e^{2t}$ is also a solution

$$\frac{d^2}{dt^2}\left[C_1 e^{-t} + C_2 e^{2t}\right] - \frac{d}{dt}\left[C_1 e^{-t} + C_2 e^{2t}\right] - 2\left[C_1 e^{-t} + C_2 e^{2t}\right] = 0$$

$$+C_1e^{-t} + 4C_2e^{2t} - \left(-C_1e^{-t} + 2C_2e^{2t}\right)$$
$$-2\left[C_1e^{-t} + C_2e^{2t}\right] = 0$$

$$\cancel{C_1e^{-t}} + \cancel{4C_2e^{2t}} + \cancel{C_1e^{-t}} - \cancel{2C_2e^{2t}} - \cancel{2C_1e^{-t}} - \cancel{2C_2e^{2t}}$$
$$= 0$$

$$0 = 0 \checkmark$$

linearly independent

u, v are said to be linearly independent if neither is a constant multiple of the other on some interval (α, β) *

ex

$u(t) = t$ and $v(t) = t^2$

$$\frac{u}{v} = \frac{t}{t^2} = \frac{1}{t} \quad \text{* not a constant}$$

are linearly independent

$U(t) = \sin(t) \quad V(t) = -4\sin(t)$

$$\frac{u}{v} = \frac{\sin(t)}{-4\sin(t)} = -\frac{1}{4}$$

linearly dependent

two linearly independent solution form a fundamental set of solutions

$$y = C_1 y_1 + C_2 y_2$$

Wronskian of two functions u, v,

$$W(t) = \det\begin{pmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{pmatrix} = u \cdot v' - u'v$$

$$\det\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$W(t) = 0$ u, v linearly dependent

$W(t) \neq 0$ u, v linearly independent

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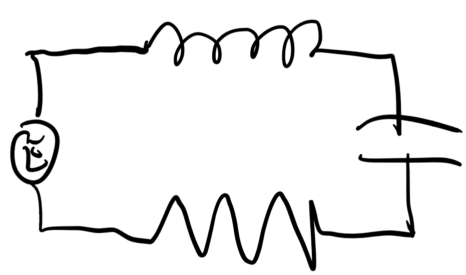
linear homogeneous Equations with constant coefficients

$$y'' + p(t)\,y' + q(t)\,y = 0$$

$$y'' + p\,y' + q\,y = 0 \qquad \begin{matrix} p \geq 0 \\ q > 0 \end{matrix} \text{ constants}$$

$$I'' + \frac{R}{L} I' + \frac{1}{LC} I = 0 \quad \text{RLC equation}$$



$$y'' + \mu\,y' + \frac{k}{m}\,y = 0 \quad \text{unforced vibrating spring}$$