Intro to Differential Equations

8/21/20

- First office hour Tues 8/25 1:30-2:30pm
  Same zoom link, same bat channel
- HW #0 Due Mon 8/24 at midnight EST (canvas + website)
- HW#1 Due Frid 8/28 at midnight EST
  1.1 # 1, 6, 7, 11
  2.1 # 1, 3, 5, 12, 13, 15

HW upload to Canvas

Mathematical modeling:

- physics
- Chemistry
- biology
- finance
- econ
- meteorology
- ecology

Two parts:

- Derivative of quantinty changing

$$\left(\frac{dx}{dt}, \frac{dv}{dt}, \frac{d\rho}{dt}\right)$$

$$(\dot{x}, \dot{v}, \dot{\rho})$$

- computing rate of change

$$\boxed{\frac{dv}{dt} = f(v, t, \ldots)}$$

Newton's Second Law

- a force acting on a mass is equal to the rate of change of momentom with respect to time

momentum $p = mv$ $\qquad$ m mass, v velocity

constant m

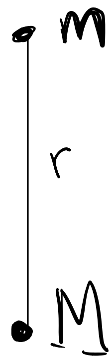
$$F = \frac{dp}{dt}$$

$$= \frac{d}{dt}(mv)$$

$$= m\frac{dv}{dt}$$

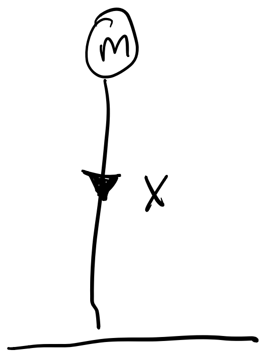
$$F = ma$$

a = acceleration



proprotional to the product of the two masses & inversely prop. to the square of the distance between them

$$|F| = \frac{GMm}{r^2}$$



$$v = \frac{dx}{dt}$$

Velocity changes in time, dependent on position

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$F = -mg \quad \text{gravity}$$

$$F = ma$$

$$ma = -mg$$

$$\cancel{m}\frac{d^2x}{dt^2} = -\cancel{m}g$$

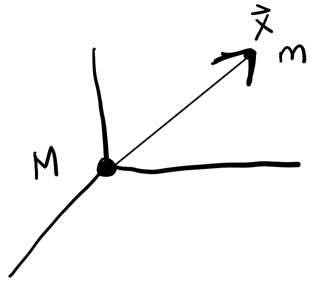
$$\frac{d^2x}{dt^2} = -g$$

- unknown function $x(t)$
- at least 1 of its derivatives

"Second order differential equation"

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} = f(x)$$

Ex 2 Planetary Motion



sun fixed at origin M

planet is at $\vec{x}(t)$ m

$$V(t) = \frac{d\vec{x}(t)}{dt}$$

$$m\frac{d^2\vec{x}}{dt} = \underbrace{-\frac{GMm}{|\vec{x}|^2}}_{\text{function of } x} \frac{\vec{x}}{|\vec{x}|}$$

unit vector in direction $\vec{x}$

2nd derivative of x

recall : $\dfrac{d^2x}{dt^2} = -g$ $\qquad x^0$

Ex Population model

Population $P(t)$, change in time

2 things :

- rate of change of Population in time

$$\boxed{\frac{dP(t)}{dt}}$$

- How the population changes

use : Biology

rate of change of a Population is (often) proportional to the population

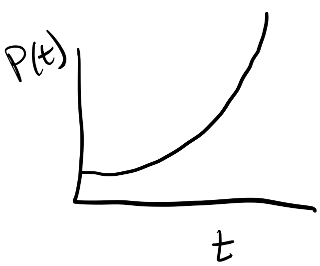
$$\boxed{rP}$$

$$\frac{dP}{dt} = rP$$

1st derivative of P — function of P

Solution to a differential equation satisfies both sides of the equation

P is a function, if plug in, both sides are the same.



$$P(t) = P_0 e^{rt}$$

↑ initial population

Check if solves ~~of~~ our Diff Eq

$$\frac{dP}{dt} = rP$$

$$\frac{d}{dt}\left[P_0 e^{rt}\right] = rP_0 e^{rt}$$

chain rule

$$P_0 \frac{d}{dt} e^{rt} = rP_0 e^{rt}$$

$$P_0 r e^{rt} = rP_0 e^{rt} \checkmark$$

Better model: $r \longrightarrow r\left(1-\frac{P}{K}\right)$

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P$$

"Logistic equation"

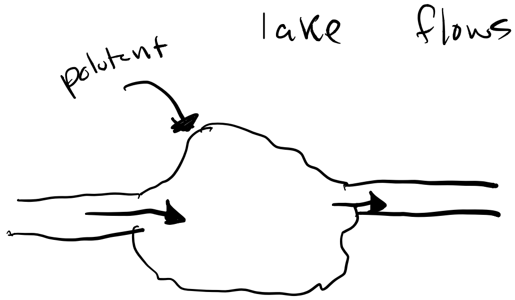
Ex Pollution

lake $V = 100\ km^3$

river flow into lake

lake flows out to another river

Volume remain constant



Flow of input ~~put~~ river depends on season (time in years)

$$\boxed{r(t) = 50 + 20\cos\left(2\pi\left(t-\tfrac{1}{4}\right)\right)}$$

$t=0$ Jan 1

$$2\pi\left(t-\tfrac{1}{4}\right) = 0$$

$$t = \tfrac{1}{4}$$ ~~month of~~

April max flow

polutant at a rate of $\boxed{2\,km^3/year}$

$X(t)$ total pollution in lake at time t.

<u>assume well mixed</u>

rate of change of pollution = amount pollution coming in − Amount pollution going out

$$\frac{dx(t)}{dt} = 2 - \left(\begin{array}{c}\text{Amount of}\\ \text{liquid flowing}\\ \text{out}\end{array}\right)\left(\begin{array}{c}\text{\% of fluid}\\ \text{which is}\\ \text{polluted}\end{array}\right)$$

$$\frac{dx(t)}{dt} = 2 - \left(50 + 20\cos\left(2\pi\left(t - \tfrac{1}{4}\right)\right)\right)\left(\%\ \text{polluted}\right)$$

$$\frac{\text{Volume polluted}}{\text{total Volume}} = \frac{x(t)}{100}$$

$$\boxed{\frac{dx(t)}{dt} = 2 - \left(50 + 20\cos\left(2\pi\left(t - \tfrac{1}{4}\right)\right)\right)\frac{x(t)}{100}}$$

derivative

function x, t