Differential Equations

HW #1 Due Friday @ midnight on canvas (corrections uploaded)

Office hours tomorrow 3-5pm EST
Zoom or Slack

Add/Drop Sept 4th

$$\frac{d}{dt}(ct) = (ct)^2$$

$$c \neq c^2 t^2$$

$$\frac{d}{dt}(2t) = (2t)^2$$

$$2 \neq 4t^2$$

Normal forms:

$$y^n = f(t, y \ldots y^{n-1})$$

$$\boxed{y' = f(t, y)}$$

Numerical Methods

approximations

some "error" in solution

IVP

$$y' = f(t, y) \quad y(a) = y_a$$

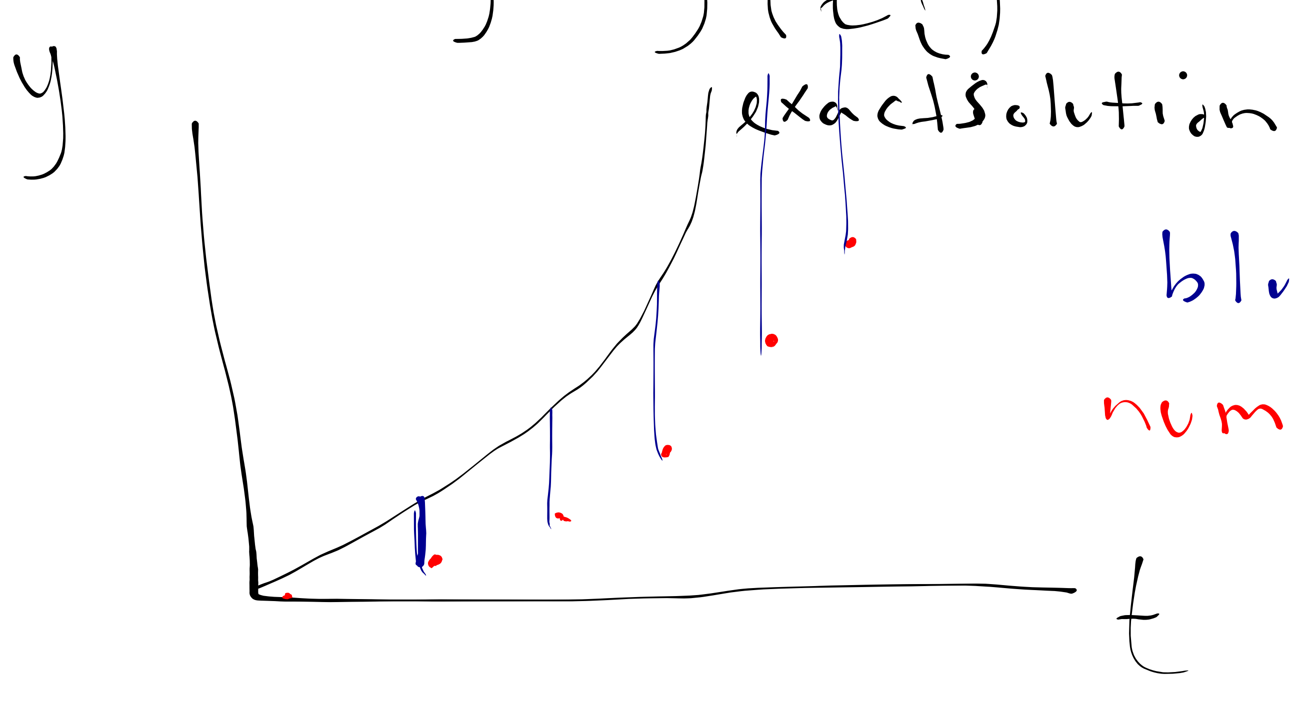
interval $a \leq t \leq b$

discrete set of points

$$a = t_0 < t_1 < t_2 < \cdots < t_N = b$$

$$y_0, y_1, y_2, \ldots \quad y_N$$

$$y = y(t_i)$$



blue error

numerical solution

Multiple methods for numerical approx.

- more accurate $\rightarrow$ complex
- quicker $\rightarrow$ less accurate
- type of solution needs a better method.

Euler's Method

Fixed step solver

chosing discrete set of values of independ variable

N sub intervals of equal size $h = \frac{b-a}{N}$ $\quad t_0 = a$

$t_N = b$

$t_1 = t_0 + h = a + h$

$t_2 = t_1 + h = a + 2h$

$\vdots$

$t_N = t_{N-1} + h = \boxed{a + Nh = b}$

$y'(t) = f(y,t)$

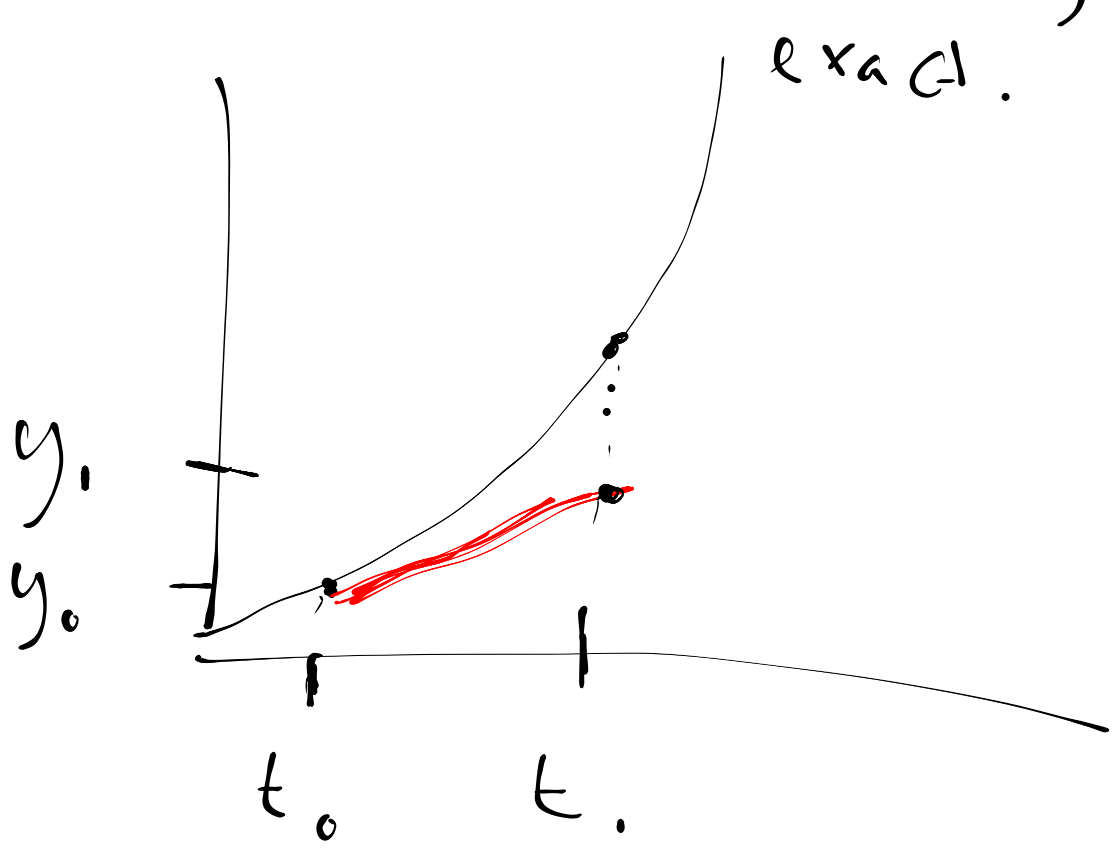
Slope at different point

→ tangent lines to guess solution

$y(t_0) + y'(t_0)h$

$y_1 = y_0 + f(t_0, y_0) \cdot h$

$t_1 = t_0 + h$



$y_2 = y_1 + f(t_1, y_1) \cdot h$

$t_2 = t_1 + h$

only depends on previous step

Single-step solver

$y_k = y_{k-1} + f(t_{k-1}, y_{k-1}) h$

$t_k = t_{k-1} + h$

Ex $\quad y' = y - t \qquad y(1) = 1$

$h = 0.1$ first few steps:

$t_0 = 1 \qquad y_0 = 1$

$$\boxed{y_1 = y_0 + f(t_0, y_0)h}$$

$$= 1 + (y_0 - t_0)h$$

$$= 1 + (1 - 1)(0.1)$$

$$y_1 = 1$$

$$\boxed{*t_1 = t_0 + h = 1 + 0.1 = 1.1}$$

$$(t_1, y_1) = (1.1, 1)$$

$$y_2 = y_1 + f(t_1, y_1)h$$

$$= y_1 + [y_1 - t_1]h$$

$$= 1 + [1 - 1.1](0.1)$$

$$= 0.99$$

$$t_2 = t_1 + h = 1.1 + 0.1 = 1.2$$

$$(t_2, y_2) = (1.2, 0.99)$$

$y' = y - t$ $\boxed{y(t) = 1 - e^{t-1}}$

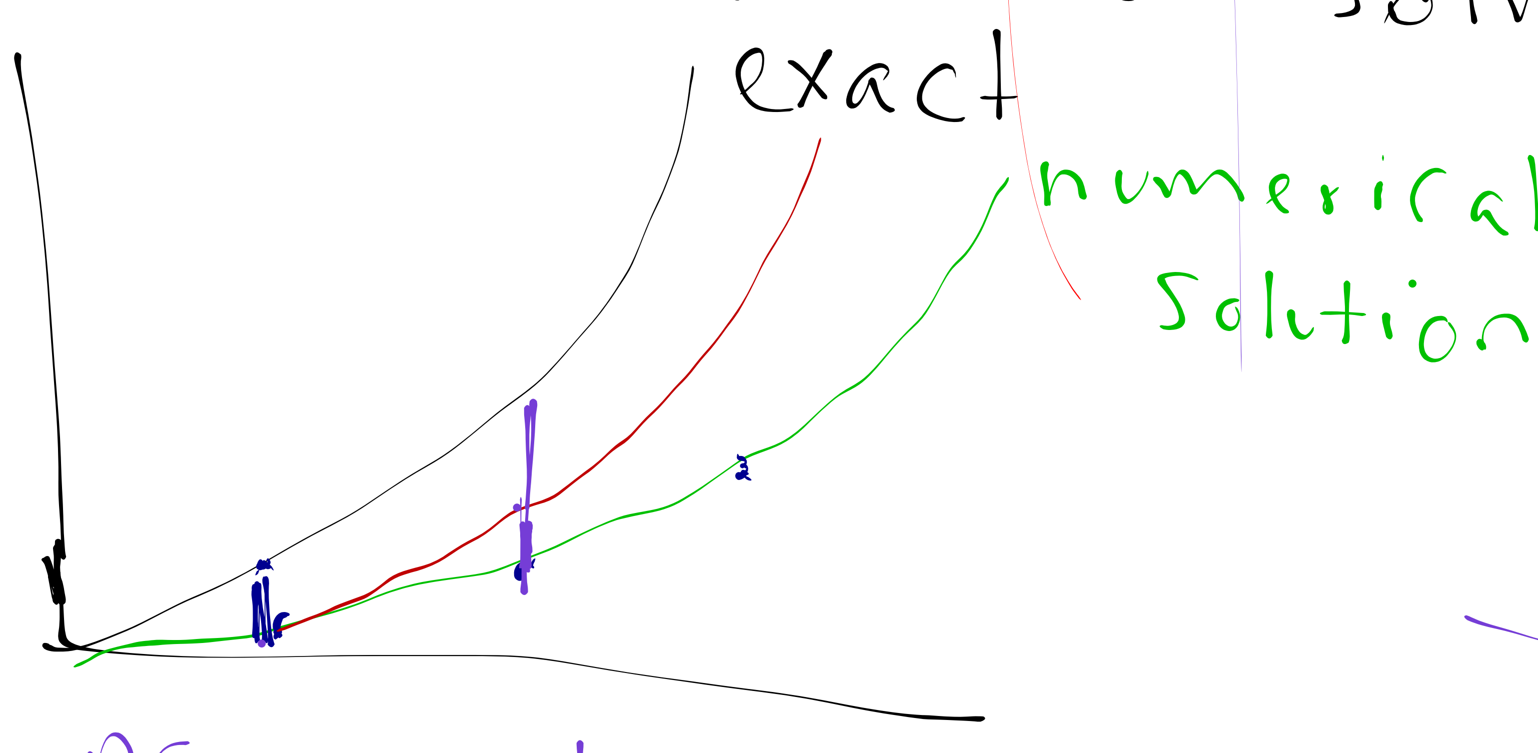
$y(t_k)$ exact solution

t_k	y_k Euler	$y(t_k)$ Exact by evaluating $y(t)$	$y(t_k) - y_k$
1.0	1.0000	1.0000	0.0000
1.1	1.0000	0.9948	-0.0052
1.2	0.9900	0.9786	-0.0114

error increase with step size

Truncation error

how accurate is our solver



propagated error

$$\text{maximum error} \leq \frac{M}{L}\left(e^{L(b-a)} - 1\right)h$$

h^2